The Problem of
Differential Calculus on Quantum Groups

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Abstract

The bicovariant differential calculi on quantum groups of Woronowicz have the drawback that their dimensions do not agree with that of the corresponding classical calculus. In this paper we discuss the first-order differential calculus which arises from a simple quantum Lie algebra $\mathfrak{l}_h(\mathfrak{g})$. This calculus has the correct dimension and is shown to be bicovariant and complete. But it does *not* satisfy the Leibniz rule. For sl_n this approach leads to a differential calculus which satisfies a simple generalization of the Leibniz rule.

1   Introduction

In my talk at this colloquium I have reviewed the definition of quantum Lie algebras $\mathfrak{l}_h(\mathfrak{g})$ inside the quantized enveloping algebras $U_h(\mathfrak{g})$ [1] given in [3]: $\mathfrak{l}_h(\mathfrak{g})$ is an indecomposable adjoint submodule of $U_h(\mathfrak{g})$ with the same dimension as $\mathfrak{g}$ on which a quantum Lie product is defined through the adjoint action. I highlighted some of the properties of these quantum Lie algebras most of which can be found in [3]-[7]. Among the most interesting is the fact that their quantum Lie product obeys a generalization of antisymmetry which involves the q-conjugation $q \leftrightarrow 1/q$.

Instead of reiterating the contents of my talk which have already been published, I want to use these proceedings to discuss a question which was raised during the colloquium. As is well known, classically the bicovariant differential calculus on a group manifold G is determined through its Lie algebra $\mathfrak{g}$, i.e., through the space of tangent vectors at the identity of G. The question was: does a quantum Lie algebras $\mathfrak{l}_h(\mathfrak{g})$ likewise determine a bicovariant differential calculus on a quantum group and what does it look like? Does it shine any light on the well known problem

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[1] We use Drinfeld's definition of $U_h(\mathfrak{g})$ as an algebra over $\mathbb{C}[[h]]$.

that a bicovariant differential calculus on a quantum group constructed according to the framework of Woronowicz does not have the correct classical dimension?

Here we show how to define the first-order differential calculus on a quantum group starting from our knowledge of its quantum Lie algebra $\mathfrak{l}_h(\mathfrak{g})$. The resulting calculus automatically has the right dimension, is complete and bicovariant. Unlike Woronowicz we do not impose the Leibniz rule. This is in the spirit of Faddeev's old idea that the Leibniz rule should be modified. In our approach the action of the differential operator d on a product is something which can be calculated and it is indeed found *not* to satisfy the standard Leibniz rule. Rather it depends on the form which the $U_h(\mathfrak{g})$ coproduct takes on $\mathfrak{l}_h(\mathfrak{g})$.

For $\mathfrak{sl}_n$ Sudbery [11] has shown that one can choose a particular $\mathfrak{l}_h(\mathfrak{sl}_n)$ on which the $U_h(\mathfrak{g})$ coproduct takes the rather simple form (3.5). The differential calculus which we construct from this quantum Lie algebra satisfies a very simple generalization (4.12) of the Leibniz rule while of course maintaining the correct dimension. A calculus of the correct dimension with a modified Leibniz rule has also been proposed by Faddeev and Pyatov [8]. The relation between these calculi should be studied. It is possible that they agree which would be interesting because our construction is so very simple and straightforward.

We expect that the results of this paper on the existence and bicovariance of the first-order differential calculus arising from our quantum Lie algebras will seem trivial to the experts of the field. We nevertheless hope that this modest contribution will be stimulating to the reader.

2 Preliminaries on quantum groups

In the theory of quantum groups one does not actually quantize the group manifold G but rather the algebra $\mathcal{F}(G)$ of regular functions on the group manifold. Classically $\mathcal{F}(G)$ is the Hopf dual of $U(\mathfrak{g})$, the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G. This means that the algebra of functions is spanned by the matrix elements of all the finite-dimensional irreducible representations of $U(\mathfrak{g})$. We define the quantized algebra of functions $\mathcal{F}_h(G)$ similarly as the Hopf dual of the quantized enveloping algebra $U_h(\mathfrak{g})$. Here $U_h(\mathfrak{g})$ is the algebra over $\mathbb{C}[[h]]$ introduced by Drinfel'd.

Definition 2.1. Let G be a connected, simply-connected, finite-dimensional, complex simple Lie group with Lie algebra $\mathfrak{g}$. The *quantized algebra of functions $\mathcal{F}_h(G)$* on G is the Hopf algebra dual of the quantized enveloping algebra $U_h(\mathfrak{g})$ in the following sense. Let $\{V^\mu\}_{\mu \in D_+}$ be the set of all finite-dimensional indecomposable $U_h(\mathfrak{g})$-modules. Choose bases $\{v^a_\mu\}_{a=1,\cdots,\dim V^\mu}$ for V^μ and introduce representation functionals $\pi^{\mu\,b}_a : U_h(\mathfrak{g}) \to \mathbb{C}[[h]]$ by[2]

$$x\, v^b_\mu = v^a_\mu\, \pi^{\mu\,b}_a(x), \quad \forall x \in U_h(\mathfrak{g}). \tag{2.1}$$

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[2]We use the summation convention according to which repeated indices are summed over.

These functionals span $\mathcal{F}_h(G)$,

$$\mathcal{F}_h(G) = \mathbb{C}[[h]]\langle\{\pi_a^{\mu b}\}\rangle_{\mu \in D_+; a, b = 1 \ldots \dim V^\mu}. \tag{2.2}$$

The coalgebra structure of $\mathcal{F}_h(G)$ is given by the algebra structure of $U_h(\mathfrak{g})$,

$$\Delta(\pi_a^{\mu b})(x \otimes y) \equiv \pi_a^{\mu b}(x \cdot y) = (\pi_a^{\mu c} \otimes \pi_c^{\mu b})(x \otimes y), \tag{2.3}$$

$$\epsilon(\pi_a^{\mu b}) \equiv \pi_a^{\mu b}(I) = \delta_a{}^b. \tag{2.4}$$

Similarly the algebra structure of $\mathcal{F}_h(G)$ is determined by the coalgebra structure of $U_h(\mathfrak{g})$ leading to an expression of the product on $\mathcal{F}_h(G)$ in terms of the $U_h(\mathfrak{g})$ Clebsch- Gordan coefficients,

$$
\begin{aligned}
(\pi_a^{\mu b} \cdot \pi_c^{\nu d})(x) &\equiv (\pi_a^{\mu b} \otimes \pi_c^{\nu d})(\Delta(x)) \\
&= \sum_\lambda (K_\lambda^{\mu\nu})_{ac}{}^i (K_{\mu\nu}^\lambda)_j{}^{bd} \, \pi_i^{\lambda j}(x),
\end{aligned} \tag{2.5}
$$

Here $K_{\mu\nu}^\lambda$ is the matrix of Clebsch-Gordan coefficients for the decompsition of $V^\mu \otimes V^\nu$ into V^λ and $K_\lambda^{\mu\nu}$ is its inverse. The second equality above is simply the defining relation for Clebsch-Gordan coefficients. We suppress the multiplicity label which is necessary if the embedding of V^λ into $V^\mu \otimes V^\nu$ is not unique.

Equation (2.5) also shows that in those cases where all irreducible representations π^μ can be obtained from the multiple tensor product of one fundamental representation $\pi^{\text{fund}} \equiv T$, the whole algebra $\mathcal{F}_h(G)$ is generated by the $T_a{}^b$ alone. Relations among these generators are obtained from the universal R-matrix R of $U_h(\mathfrak{g})$,

$$
\begin{aligned}
(T_a{}^b \cdot T_c{}^d)(x) &= (T_a{}^b \otimes T_c{}^d)(\Delta(x)) \\
&= (T_a{}^b \otimes T_c{}^d)(R^{-1}\Delta^{\text{op}}(x)\,R) \\
&= ((R^{-1})_{ac}{}^{a'c'} T_{c'}{}^{d'} \cdot T_{a'}{}^{b'} R_{b'd'}{}^{bd})(x),
\end{aligned} \tag{2.6}
$$

where $R_{ab}{}^{cd} = (T_a{}^c \otimes T_b{}^d)R$, etc. Additional relations express the linear dependence among the $T_a{}^b$. These are the relations of the FRT algebras of Faddeev, Reshetikhin and Takhtajan [9]. The FRT algebras are defined only for $G = SL_n(\mathbb{C}), SO_n(\mathbb{C})$ and $Sp_n(\mathbb{C})$. It is believed that for $G = SL_n(\mathbb{C})$ and $Sp_n(\mathbb{C})$ the FRT algebras coincide with $\mathcal{F}_h(G)$ defined in Definition 2.1. This can not be the case for the groups $SO_n(\mathbb{C})$ because they are not simply connected. Because in this paper we are interested in constructing the differential geometry on the group manifold by group translation from the tangent vector at the origin, we work with the quantized function algebra on their simply connected covering groups as defined in Definition 2.1 rather than with the FRT algebras.

3 Quantum Lie algebras

We will be very brief here, see [3]-[7] for details.

A Lie algebra $\mathfrak{g}$ carries the adjoint representation of $U(\mathfrak{g})$. Let π^Ψ denote the adjoint representation of $U_h(\mathfrak{g})$ (this is the unique indecomposable representation of $U_h(\mathfrak{g})$ with dimension equal to dim $\mathfrak{g}$ and is a deformation of the classical adjoint representation).

Definition 3.1. A *weak quantum Lie algebra* $\mathfrak{l}_h(\mathfrak{g})$ is a submodule of $U_h(\mathfrak{g})$ which generates $U_h(\mathfrak{g})$ and on which the adjoint action gives the adjoint representation. The *quantum Lie product* is given by $[a,b]_h = (\operatorname{ad} a)\, b$ for all $a, b \in \mathfrak{l}_h(\mathfrak{g})$.

This is a reformulation of the definition given in [3] making use of the understanding gained in [5]. *Strong* quantum Lie algebras $\mathfrak{L}_h(\mathfrak{g})$ have additional properties regarding the action of the antipode, the Cartan involution and any diagram automorphism, see [3].

By the statement that the adjoint action on $\mathfrak{l}_h(\mathfrak{g})$ gives the adjoint representation we mean that $\mathfrak{l}_h(\mathfrak{g})$ has a basis $\{t^i\}_{i=1,\cdots \dim \mathfrak{g}}$ which satisfies

$$(\operatorname{ad} x)\, t^i = t^j\, \pi_j^{\Psi\, i}(x), \qquad \forall x \in U_h(\mathfrak{g}). \tag{3.1}$$

When we say that $\mathfrak{l}_h(\mathfrak{g})$ generates $U_h(\mathfrak{g})$ we mean that any element of $U_h(\mathfrak{g})$ can be expressed as a formal power series in h whose coefficients are polynomials in the t^i.

The representation matrices of the quantum Lie algebra generators t^i are given by Clebsch-Gordan coefficients,

$$\pi_a^{\mu b}(t^i) = (K_{\mu\Psi}^\mu)_a{}^{bi}, \qquad \forall \mu \in D_+, \tag{3.2}$$

This can be derived by applying $\pi_a^{\mu b}$ to the equality[3] $x\, t^i = ((\operatorname{ad} x_{(2)})\, t^i)\, x_{(1)}$ and recognizing the result as the intertwining property of the Clebsch-Gordan coefficients. This implies in particular that the structure constants of the quantum Lie algebra are given by the Clebsch-Gordan coefficients for adjoint $\otimes$ adjoint into adjoint,

$$[t^i, t^j] = (K_{\Psi\Psi}^\Psi)_k{}^{ji}\, t^k. \tag{3.3}$$

For more details see [5].

Inside any $U_h(\mathfrak{g})$ there exist infinitely many quantum Lie algebras $\mathfrak{l}_h(\mathfrak{g})$, see [5] for their construction. However the equations $(3.2), (3.3)$ hold for any of them, the only differences being in the choice of K. In particular, except for $\mathfrak{g} = \mathfrak{sl}_n$ with $n > 2$, all quantum Lie algebras $\mathfrak{l}_h(\mathfrak{g})$ inside $U_h(\mathfrak{g})$ are isomorphic. This is due to the uniqueness of the embedding of the adjoint representation into adjoint $\otimes$ adjoint. In the case of $\mathfrak{g} \neq \mathfrak{sl}_n$ with $n > 2$ uniqueness holds after imposing the extra requirement of invariance under the diagram automorphism of $U_h(\mathfrak{sl}_n)$. Because all

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[3]Here we use the definition of the adjoint action $(\operatorname{ad} x)y = x_{(2)}\, y\, S^{-1}(x_{(1)})$.

$\mathfrak{l}_h(\mathfrak{g})$ are isomorphic we view them as different embeddings of an abstract quantum Lie algebra $\mathfrak{g}_h$ into $U_h(\mathfrak{g})$.

The fact that there exist infinitely many isomorphic quantum Lie algebras inside $U_h(\mathfrak{g})$ is not surprising. Also classically there are infinitely many embeddings of a Lie algebra $\mathfrak{g}$ into the enveloping algebra $U(\mathfrak{g})$ (multiplying all elements of one embedding by a central element of $U_h(\mathfrak{g})$ gives a new embedding). However, among these there is a unique embedding, which we will denote by $\mathfrak{l}^0(\mathfrak{g})$, with the property that $\mathfrak{l}^0(\mathfrak{g})$ generates $U(\mathfrak{g})$. (This is the natural embedding and therefore commonly $\mathfrak{l}^0(\mathfrak{g})$ would be denoted simply as $\mathfrak{g}$). The requirement that $\mathfrak{l}_h(\mathfrak{g})$ generate $U_h(\mathfrak{g})$ implies that $\mathfrak{l}_h(\mathfrak{g}) = \mathfrak{l}^0(\mathfrak{g}) \mod h$. However this requirements does not put any restrictions on $\mathfrak{l}_h(\mathfrak{g})$ at higher orders in h.

Classically $\mathfrak{l}^0(\mathfrak{g})$ is also special among all embeddings $\mathfrak{l}(\mathfrak{g})$ because on $\mathfrak{l}^0(\mathfrak{g})$ the coproduct of $U(\mathfrak{g})$ takes the particularly simple form

$$\Delta(a) = 1 \otimes a + a \otimes 1, \quad \forall a \in \mathfrak{l}^0(\mathfrak{g}). \tag{3.4}$$

We expect that also at the quantum level some $\mathfrak{l}_h(\mathfrak{g})$ will be singled out by their property under the $U_h(\mathfrak{g})$ coproduct. This property should be some quantum generalization of (3.4). Unfortunately the correct generalization has so far not been found in general. For $\mathfrak{g} = \mathfrak{sl}_n$ Sudbery [11] has shown that one can choose a $\mathfrak{l}_h^0$ which satisfies the following generalization of (3.4)

$$\Delta(t^i) = C \otimes t^i + t^j \otimes f_j{}^i, \tag{3.5}$$

where f_i^j are certain elements in $U_h(\mathfrak{g})$ and C is a central element of $U_h(\mathfrak{g})$. We will see the relevance of this for the Leibniz rule in the quantum differential calculus later.

4 First-order differential calculus

For background on differential calculus on quantum groups I recommend the pioneering paper by Woronowicz [12] and the review [1]. We take vector fields to be the starting point rather than forms. Such an approach has been studied in [2].

A quantum Lie algebra $\mathfrak{l}_h(\mathfrak{g})$ is interpreted as the quantum analog of the space of tangent vectors to the group manifold at the identity. By left-translating a tangent vector t one obtains a left-invariant vector field t_L. In order to be applied at the quantum level, the notion of left-translation has to be formulated in terms of the functions on the group manifold rather than the points. As explained in [12, 1] this leads to

$$t_L(a) = a_{(1)} \, t(a_{(2)}), \qquad a \in \mathcal{F}_h(G), \ t \in \mathfrak{l}_h(\mathfrak{g}). \tag{4.1}$$

Here the action of t on $a_{(2)}$ is given by the duality between $U_h(\mathfrak{g})$ and $\mathcal{F}_h(G)$, i.e., $t(a_{(2)}) = a_{(2)}(t)$. In this way the quantum Lie algebra generators t^i lead to basis

vectors t_L^i for the space[4] of left-invariant vector fields, $\mathcal{V}_L = \mathbb{C}[[h]]\langle\{t_L^i\}\rangle$.

Classically an arbitrary vector field can be written as a linear combination of products of left-invariant vector fields by functions. In the quantum case we define the product $(t_L a)$ of a left-invariant vector fields t_L by an element a of $\mathcal{F}_h$ from the right by $(t_L a)(b) = t_L(b)\, a$.

Definition 4.1. The space $\mathcal{V}$ of *vector fields* on the quantum group is the right $\mathcal{F}_h$-module freely generated by $\{t_L^i\}$, $\mathcal{V} = \langle\{t_L^i\}\rangle \mathcal{F}_h$.

The reason why we choose to multiply vector fields by functions from the right is that then one-forms will be multiplied from the left, as we will see below. Classically there is of course no distinction.

We could alternatively start with the right invariant-vector fields t^R obtained by right translating the Lie algebra elements, giving $t^R(a) = t(a_{(1)})\, a_{(2)}$, $a \in \mathcal{F}_h(G)$, $t \in \mathfrak{l}_h(\mathfrak{g})$. We could use these to define the space of all vector fields as $\mathcal{V} = \langle\{t_R^i\}\rangle \mathcal{F}_h$. Our formalism would not be natural if this lead to a different notion of vector fields. Luckily we have

Proposition 4.1. *Left- and right-invariant vector fields are related by*

$$t_L^i = t_R^j \pi_j^{\Psi i}. \tag{4.2}$$

Proof. We need to show that $t_L^i(\pi_a^{\mu c}) = t_R^j(\pi_a^{\mu c})\, \pi_j^{\Psi i}$, $\forall \pi_a^{\mu c}$. Using the definitions of t_L^i (4.1) and t_R^i and the formula (2.3) for the coproduct this becomes $\pi_a^{\mu b} t^i(\pi_b^{\mu c}) = t^j(\pi_a^{\mu b})\, \pi_b^{\mu c}\, \pi_j^{\Psi i}$ which holds because of (3.2) and the intertwining property of the Clebsch-Gordan coefficients. $\qquad\qquad\square$

Conversely we have $t_R^i = t_L^j\, ((\pi^{\Psi})^{-1})_j{}^i$ and as a consequence $\mathcal{V} = \langle\{t_L^i\}\rangle \mathcal{F}_h = \langle\{t_R^i\}\rangle \mathcal{F}_h$. We note that it was not a priori obvious that this would work. In order to have the same dimension as classically, we have dropped some of the left-invariant vector fields t_L^i which appear in the Woronowicz calculus. It could have happened that the expression of the right-invariant vector fields in terms of the left-invariant ones would have reintroduced the ones we had dropped and our truncation would not have been consistent.

We now introduce the space Γ_L of left-invariant one-forms as the dual space to the space $\mathcal{V}_L$ of left-invariant vector fields and we choose a basis $\{\omega_i^L\}$ such that $\omega_i^L(t_L^j) = \delta_i{}^j$. We define the action of a $\omega^L \in \Gamma$ on an arbitrary vector field by setting $\omega^L(t_L\, a) = (\omega^L(t_L))a$, $\forall t_L \in \mathcal{V}_L, a \in \mathcal{F}_h$. We define the left multiplication of one-forms with functions by $(a\,\omega)(t) = a(\omega(t))$, $\forall t \in \mathcal{V}, a \in \mathcal{F}_h$.

Definition 4.2. The space Γ of *one-forms* on the quantum group is the left $\mathcal{F}_h$-module freely generated by $\{\omega_i^L\}$, $\Gamma = \mathcal{F}_h\langle\{\omega_i^L\}\rangle$.

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[4] Here we use the word "space" even when the base is only a ring, the term "free module" would be more precise.

Again we could equivalently have used right-invariant one-forms ω_i^R satisfying $\omega_i^R(t_R^j) = \delta_i{}^j$. These are related to the left-invariant ω_i^L by

$$\omega_i^R = \pi_i^{\Psi j}\omega_j^L, \tag{4.3}$$

and thus $\Gamma = \mathcal{F}_h\langle\{\omega_i^L\}\rangle = \mathcal{F}_h\langle\{\omega_i^R\}\rangle$.

As explained in [12, 1], left- and right-translation of one-forms is described by maps $\Delta_L : \Gamma \to \mathcal{F}_h \otimes \Gamma$ and $\Delta_R : \Gamma \to \Gamma \otimes \mathcal{F}_h$. They act as

$$\Delta_L(a\,\omega^L) = \Delta(a)\,(1 \otimes \omega^L), \qquad \Delta_R(a\,\omega^R) = \Delta(a)\,(\omega^R \otimes 1). \tag{4.4}$$

Because any one-form can be expressed in terms of left- or right-invariant forms, this determines left- and right-translation entirely. Because of the way in which we have defined the left- and right- translations the bicovariance condition $(1 \otimes \Delta_R)\Delta_L = (\Delta_L \otimes 1)\Delta_R$ is guaranteed. Of course it can also be checked by explicit calculation.

Definition 4.3. The *exterior differential* $d : \mathcal{F}_h \to \Gamma$ is defined by

$$da(v) = v(a) \qquad \forall a \in \mathcal{F}_h,\, v \in \mathcal{V}. \tag{4.5}$$

In terms of the bases this can be expressed as

$$da = t_L^i(a)\,\omega_i^L = t_R^i(a)\,\omega_i^R. \tag{4.6}$$

Proposition 4.2. *The first-order differential calculus* (Γ, d) *satisfies*

(i) *Completeness: any $\rho \in \Gamma$ can be written as $\rho = \sum_k a_k\, d\, b_k$ for some $a_k, b_k \in \mathcal{F}_h$.*

(ii) *Bicovariance: for all $a, b \in \mathcal{F}_h$,*

$$\Delta_L(a\, db) = \Delta(a)\,(1 \otimes d)\,\Delta(b), \qquad \Delta_R(a\, db) = \Delta(a)\,(d \otimes 1)\,\Delta(b). \tag{4.7}$$

Proof. i) Here we can follow [1]. We will show that any ω_L^i can be written in this form. Define the matrix $\gamma^{ij} = \pi_a^{\Psi b}(t^i)\,\pi_b^{\Psi a}(t^j)$. Classically γ goes over into the Killing metric and thus γ is invertible also in the quantum case. It can now be checked that[5]

$$\omega_i^L = (\gamma^{-1})_{ij}\pi_a^{\Psi b}(t^j)((\pi^\Psi)^{-1})_b{}^c\, d\,\pi_c^{\Psi a}. \tag{4.8}$$

ii)by a simple calculation: $\Delta_L(a\, db) = \Delta(a)\Delta_L(t_L^i(b)\omega_i^L) = \Delta(a)\Delta(b_{(1)})t^i(b_{(2)})(1 \otimes \omega_i^L) = \Delta(a)(b_{(1)} \otimes b_{(2)}t^i(b_{(3)})\omega_i^L) = \Delta(a)(b_{(1)} \otimes db_{(2)}) = \Delta(a)(1 \otimes d)\Delta(b)$. $\qquad\square$

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[5] Remember that $t^i(\pi_a^{\Psi b}) \equiv \pi_a^{\Psi b}(t^i)$.

Woronowicz, in his approach to bicovariant quantum differential calculus, postulates completeness and bicovariance as axioms rather than deriving them. He postulates one more axiom: the Leibniz rule

$$d(ab) = a(db) + (da)b, \qquad \forall\, a, b \in \mathcal{F}_h \tag{4.9}$$

In the second term this equation involves multiplication of a one-form by a function from the right. It thus makes sense only if Γ is a $\mathcal{F}_h$-bimodule. The Leibniz rule is the only reason why one is interested in having a bimodule structure on Γ. It is introduced by defining the right multiplication of one-forms by functions through

$$\omega_i^L\, a = a_{(1)}\, f_i{}^j(a_{(2)})\, \omega_j^L, \qquad \forall\, a \in \mathcal{F}_h, \tag{4.10}$$

where $f^i{}_j$ are certain elements of $U_h(\mathfrak{g})$.

In our approach the Leibniz rule is not an axiom but rather the action of d on a product of functions can be computed in terms of the known coproduct $\Delta(t^i) = t^i_{(1)} \otimes t^i_{(2)}$.

$$d(ab) = a_{(1)} b_{(1)} t^i(a_{(2)} b_{(2)}) \omega_i^L = a_{(1)} b_{(1)} t^i_{(1)}(a_{(2)}) t^i_{(2)}(b_{(2)}) \omega_i^L. \tag{4.11}$$

Even though all quantum Lie algebras $\mathfrak{l}_h(\mathfrak{g})$ inside the same $U_h(\mathfrak{g})$ are isomorphic as algebras, the coproducts $\Delta(t^i)$ differ and can be very complicated in general. It is important to define the differential calculus in terms of particular quantum Lie algebras $\mathfrak{l}_h^0(\mathfrak{g})$ whose coproduct is simple, otherwise one will obtain an unmanageable Leibniz rule. For $\mathfrak{sl}_n$ a good $\mathfrak{l}_h^0(\mathfrak{sl}_n)$ has been found by Sudbery [11] which has the coproduct given in (3.5). It leads to a differential calculus with generalized Leibniz rule

$$d(ab) = c(a)(db) + (da)b, \qquad \forall\, a, b \in \mathcal{F}_h, \tag{4.12}$$

where $c(a) \equiv a_{(1)} C(a_{(2)})$ and right multiplication is defined as in (4.10).

5   Discussion

We have explained that by starting from the quantum Lie algebras $\mathfrak{l}_h(\mathfrak{g})$ as studied in [3]-[7] and [11] one automatically obtains complete, bicovariant quantum differential calculi (Γ, d) of the correct dimension which do however not satisfy the standard Leibniz rule. The straightforward construction can be summarized as follows: 1) interpret the quantum Lie algebra elements as tangent vectors at the identity, 2) left- or right-translate them to obtain left- or right-invariant vector fields, 3) take their duals to obtain left- or right invariant one-forms, 4) multiply them by functions to obtain the space Γ of all one-forms, 5) define the exterior differential as in Definition 4.3.

This paper has been kept very brief. Many properties of the differential calculus have been left unstudied, in particular the exterior calculus and the Cartan calculus.

For the later we would like to draw the reader's attention to the elegant work of Schupp and coworkers, see e.g. [10].

The big challenge which remains is to identify among the infinitely many quantum Lie algebras $\mathfrak{l}_h(\mathfrak{g})$ those with the simplest coproduct which will lead to the differential calculus with the most natural generalization of Leibniz rule. This has so far been accomplished only for $\mathfrak{sl}_n$ by Sudbery [11].

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For an extensive list of references on quantum Lie algebras and on differential calculus on quantum groups visit the World Wide Web site on quantum Lie algebras at http://www.mth.kcl.ac.uk/~delius/q-lie.html